

EQUILIBRIUM MEASURES FOR UNIFORMLY QUASIREGULAR DYNAMICS

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ABSTRACT. We establish the existence and fundamental properties of the equilibrium measure in uniformly quasiregular dynamics. We show that a uniformly quasiregular endomorphism f of degree at least 2 on a closed Riemannian manifold admits an equilibrium measure μ_f , which is balanced and invariant under f and non-atomic, and whose support agrees with the Julia set of f . Furthermore we show that f is strongly mixing with respect to the measure μ_f . We also characterize the measure μ_f using an approximation property by iterated pullbacks of points under f up to a set of exceptional initial points of Hausdorff dimension at most $n - 1$. These dynamical mixing and approximation results are reminiscent of the Mattila-Rickman equidistribution theorem for quasiregular mappings. Our methods are based on the existence of an invariant measurable conformal structure due to Iwaniec and Martin and the \mathcal{A} -harmonic potential theory.

1. INTRODUCTION

By the classical Liouville theorem, conformal mappings on the standard n -sphere \mathbb{S}^n are Möbius transformations for $n \geq 3$. Quite surprisingly, Iwaniec and Martin showed in [11] that there exist measurable conformal structures on the n -sphere \mathbb{S}^n admitting conformal mappings that are not local homeomorphisms. These measurable conformal structures are quasiconformally equivalent to the standard conformal structure on \mathbb{S}^n and these conformal mappings, that plays the rôle in rational maps, are *uniformly quasiregular* with respect to the standard structure.

Recall that a continuous mapping $f: \mathbb{M} \rightarrow \mathbb{M}$ on an oriented Riemannian n -manifold \mathbb{M} is *K -quasiregular*, for some $K \geq 1$, if f belongs to the local Sobolev space $W_{\text{loc}}^{1,n}(\mathbb{M}, \mathbb{M})$ and satisfies the distortion inequality

$$(1.1) \quad \|Df\|^n \leq K J_f \quad \text{a.e. } \mathbb{M},$$

where $\|Df\|$ is the point-wise operator norm of the differential Df of the mapping f and J_f is the Jacobian determinant of the differential. A quasiregular endomorphism $f: \mathbb{M} \rightarrow \mathbb{M}$ is said to be *uniformly quasiregular* if there exists $K \geq 1$ so that all the iterates of f are K -quasiregular.

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Results of Peltonen [20] and Astola, Kangaslampi, and Peltonen [1] show that the existence of branching uniformly quasiregular dynamics is not limited to \mathbb{S}^n . However, results of Martin, Meyer, and Peltonen [15] show that, among space forms, only spherical space forms admit branching uniformly quasiregular endomorphisms; see [15] for the terminology and for related results. Since the existence question of branching uniformly quasiregular dynamics is interesting also for manifolds that are not space forms, we state our results for general closed manifolds.

To state our main results, we recall that a Borel measure μ on \mathbb{M} is *balanced under f* if

$$f^*\mu = (\deg f)\mu,$$

where $f^*\mu$ is the pull-back of μ under f ; see Section 5 for details. We note that if μ is balanced under f , then it is *invariant* under f , that is, $f_*\mu = \mu$.

Our main result is the following existence theorem for an invariant measure; the approximation property of μ_f is reminiscent to the complex analytic case, see Lyubich [14].

In all of the following statements, we assume that \mathbb{M} is closed, connected, and oriented Riemannian n -manifold. Here \rightarrow means the weak convergence of measures on \mathbb{M} .

Theorem 1. *Let $f: \mathbb{M} \rightarrow \mathbb{M}$ be a uniformly quasiregular endomorphism of degree $d > 1$. Then there exists an invariant measure μ_f satisfying the approximation (or equidistribution) property*

$$\frac{(f^k)^*\delta_a}{d^k} \rightarrow \mu_f$$

as $k \rightarrow \infty$ for every $a \in \mathbb{M}$ except for a subset of Hausdorff dimension at most $n - 1$ in \mathbb{M} .

We prove Theorem 1 in two parts: first the existence in Theorem 5.2 and then the approximation (or equidistribution) property in Theorem 7.1.

We call the measure μ_f an *equilibrium measure of f* since this measure has properties familiar from complex dynamics. Here, the Julia set of f is defined by

$$\mathcal{J}(f) := \{x \in \mathbb{M}; \{f^k; k \in \mathbb{N}\} \text{ is not normal on any open ball around } x\}.$$

Theorem 2. *Let $f: \mathbb{M} \rightarrow \mathbb{M}$ be a uniformly quasiregular endomorphism of degree $d > 1$. Then the equilibrium measure μ_f is non-atomic and balanced under f , and the support $\text{spt } \mu_f$ agrees with $\mathcal{J}(f)$.*

As our final result, we show that f is mixing with respect to μ_f . This result and the approximation property with respect to μ_f in Theorem 1 can be viewed as dynamical counterparts of the Mattila-Rickman equidistribution theorem [18, Theorem 5.1].

Theorem 3. *Let $f: \mathbb{M} \rightarrow \mathbb{M}$ be a uniformly quasiregular endomorphism of degree $d > 1$. Then f is strongly mixing with respect to the invariant measure μ_f , that is, for every $\phi, \psi \in L^2(\mu_f)$,*

$$(1.2) \quad \lim_{k \rightarrow \infty} \int_{\mathbb{M}} (\phi \circ f^k) \psi d\mu_f = \int_{\mathbb{M}} \psi d\mu_f \int_{\mathbb{M}} \phi d\mu_f.$$

In particular, μ_f is ergodic under f .

The proofs of Theorems 1 and 3 are, in spirit, potential theoretic. Given a manifold \mathbb{M} and a uniformly quasiregular endomorphism f of \mathbb{M} , we interpret the invariant conformal structure of Iwaniec and Martin in terms of the given Riemannian structure. This translates the n -Laplace operator Δ_n of the measurable conformal structure to an f -invariant \mathcal{A} -harmonic equation. Although the n -Laplace operator Δ_n and the corresponding \mathcal{A} -harmonic operator are non-linear, the methods in [5] are within our reach.

2. PRELIMINARIES

Let \mathbb{M} be a closed, oriented, and connected Riemannian n -manifold, $n \geq 2$. We denote the Riemannian metric on \mathbb{M} by $\langle \cdot, \cdot \rangle$ and the induced distance between points x and y on \mathbb{M} by $|x - y|$. Given $x \in \mathbb{M}$ and $r > 0$, we denote by $B(x, r)$ the open ball of radius r about x in this metric. The corresponding closed ball is $\bar{B}(x, r)$. We denote by $\text{vol}_{\mathbb{M}}$ the volume form induced by this metric. We denote by \mathcal{H}^s the *Hausdorff s -measure* on \mathbb{M} associated to the distance.

We denote by $L^p(\wedge^m \mathbb{M})$ the space of p -integrable m -forms on \mathbb{M} and by $W^{1,p}(\wedge^m \mathbb{M})$ the Sobolev space of p -integrable m -forms having (weak) exterior derivative in $L^p(\mathbb{M})$; see [12] for details. We denote also by \star the *Hodge star* associated to the Riemannian metric and by d^* the *co-exterior derivative* on \mathbb{M} , that is, the operator $d^* = (-1)^{m(n-m)} \star d \star$ on m -forms.

Given a measure μ on \mathbb{M} , we denote the *support* of μ , i.e. the smallest closed set E satisfying $\mu(\mathbb{M} \setminus E) = 0$, by $\text{spt } \mu$.

Throughout the article we consider quasiregular endomorphisms f on closed Riemannian n -manifolds \mathbb{M} , that is, continuous mappings $\mathbb{M} \rightarrow \mathbb{M}$ in the Sobolev class $W^{1,n}(\mathbb{M}, \mathbb{M})$ satisfying the distortion inequality (1.1). We define the Sobolev space $W^{1,n}(\mathbb{M}, \mathbb{M})$ as in [6], that is, we fix a smooth Nash embedding $\iota: \mathbb{M} \rightarrow \mathbb{R}^N$, and say that $f \in W^{1,n}(\mathbb{M}, \mathbb{M})$ if coordinates of $\iota \circ f$ are Sobolev functions in $W^{1,n}(\mathbb{M})$.

The *push-forward* $f_*\psi: \mathbb{M} \rightarrow \mathbb{R}$ of $\psi \in C^0(\mathbb{M})$ by a quasiregular endomorphism f on \mathbb{M} is defined by

$$f_*\psi(y) := \sum_{x \in f^{-1}(y)} i(x, f)\psi(x),$$

where $i(x, f)$ is the *local index* of f at x ; see e.g. [22, Section 4]. In particular, $f_*\psi$ is continuous and belongs to $W^{1,n}(\mathbb{M})$; see [7, Lemmas 14.30 and 14.31]. Given a Borel measure μ on \mathbb{M} the pull-back measure $f^*\mu$ is defined by the formula

$$\int_{\mathbb{M}} \phi d(f^*\mu) = \int_{\mathbb{M}} f_*\phi d\mu,$$

where $\phi \in C^0(\mathbb{M})$.

In the proof of Theorem 1 we use both classical potential theory of measures and non-linear potential theory of \mathcal{A} -harmonic equations. We recall now some classical facts on potential theory that will be used in the following sections.

Given a finite Borel measure μ on \mathbb{M} , we denote by $U^\mu(x)$ the (Riesz) 1-potential of μ at $x \in \mathbb{M}$, that is,

$$U^\mu(x) = \int_{\mathbb{M}} \frac{d\mu(y)}{|x - y|^{n-1}}$$

for $x \in \mathbb{M}$.

Given a compact set $E \subset \mathbb{M}$, the energy functional

$$(2.1) \quad I_1(\mu) = \int_E \int_E \frac{d\mu(x)d\mu(y)}{|x - y|^{n-1}}$$

admits the unique minimizer λ_E among all probability Borel measures μ on E if $\sup_{\mu} I_1(\mu) < \infty$, where μ ranges over all probability Borel measures on E ; see e.g. [13, Chapter II §1]. We call the measure λ_E the *equilibrium measure of E with respect to the Riesz 1-capacity* and denote by $W_1(E)$ the minimal energy $I_1(\lambda_E)$. The *Riesz 1-capacity* $C_1(E)$ of the compact set E is

$$C_1(E) = \frac{1}{W_1(E)}.$$

We have

$$(2.2) \quad U^{\lambda_E}(x) \leq \frac{1}{C_1(E)}$$

for all $x \in \mathbb{M}$; in fact, $U^{\lambda_E} \equiv C_1(E)^{-1}$ in $\text{spt } \mu \setminus E'$, where $C_1(E') = 0$; see e.g. [13, p. 137] for more details.

3. \mathcal{A} -HARMONIC POTENTIALS

In this section we prove the following proposition on the existence of solutions to certain \mathcal{A} -harmonic Poisson-type equations.

Proposition 3.1. *Let $f: \mathbb{M} \rightarrow \mathbb{M}$ be a K -quasiregular endomorphism of degree $d > 0$. Then there exists $p = p(n, K) > 1$ so that for every $\omega \in L^p(\bigwedge^n \mathbb{M})$, the following \mathcal{A} -harmonic equation*

$$(3.1) \quad d^* \mathcal{A}(du) = \star \left(\frac{f^* \omega}{d} - \omega \right)$$

has a solution $u \in W^{1,n}(\mathbb{M})$.

In (3.1), \mathcal{A} is a uniformly elliptic measurable bundle map $\mathcal{A}: T^*\mathbb{M} \rightarrow T^*\mathbb{M}$ so that there exist $a > 0$ and $b > 0$ satisfying the following conditions:

- (1) $\langle \mathcal{A}(\xi) - \mathcal{A}(\zeta), \xi - \zeta \rangle \geq a(|\xi| + |\zeta|)^{n-2} |\xi - \zeta|^2$,
- (2) $|\mathcal{A}(\xi) - \mathcal{A}(\zeta)| \leq b(|\xi| + |\zeta|)^{n-2} |\xi - \zeta|$, and
- (3) $\mathcal{A}(t\xi) = t|t|^{n-2} \mathcal{A}(\xi)$

for all $\xi, \zeta \in T_x^* \mathbb{M}$, $t \in \mathbb{R}$, and for almost all $x \in \mathbb{M}$. Furthermore, we assume that for every measurable 1-form ω on \mathbb{M} , so is $x \mapsto \mathcal{A}_x(\omega)$. We refer [7] for a detailed discussion on the non-linear potential theory of \mathcal{A} -harmonic equations.

The proof of Proposition 3.1 is based on the higher integrability of quasiregular mappings and the following lemma on solvability of \mathcal{A} -harmonic Poisson equations.

Lemma 3.2. *Let $p > 1$ and let $\omega \in L^p(\bigwedge^n T\mathbb{M})$ be an n -form so that*

$$(3.2) \quad \int_{\mathbb{M}} \omega = 0.$$

Then there exists a solution $u \in W^{1,n}(\mathbb{M})$ to the \mathcal{A} -harmonic equation

$$(3.3) \quad d^* \mathcal{A}(du) = \star \omega,$$

Proof. By (3.2), ω is weakly exact and there exists an $(n-1)$ -form $\tau \in L^p(\bigwedge^{n-1} \mathbb{M})$ so that $d\tau = \omega$. Moreover, by the Poincaré inequality for differential forms, there exists a weakly closed $(n-1)$ -form τ_0 so that $\tau - \tau_0 \in W^{1,p}(\bigwedge^{n-1} \mathbb{M})$; see [12, Theorem 6.4]. Since $d(\tau - \tau_0) = d\tau = \omega$, we may assume that $\tau \in W^{1,p}(\bigwedge^{n-1} \mathbb{M})$. Since

$$d^* \mathcal{A}(du) = \star \omega = \star d\tau,$$

it suffices to show that, up to a sign, there exists u so that

$$\star \mathcal{A}(du) = \tau.$$

Since $p > 1$, the Sobolev embedding theorem concludes $\tau \in L^{p^*}(\bigwedge^{n-1} \mathbb{M})$, where $p^* := np/(n-p)$. Since $p^* > n/(n-1)$ and $\mathcal{A}: L^n(\bigwedge^1 \mathbb{M}) \rightarrow L^{n/(n-1)}(\bigwedge^1 \mathbb{M})$ is surjective by the Minty-Browder theorem, there exists a solution $u \in W^{1,n}(\mathbb{M})$; see [12, Section 8] and, e.g., [4, Sections 2.3 and 2.4] for details. \square

Proof of Proposition 3.1. By change of variables,

$$\int_{\mathbb{M}} \frac{f^* \omega}{d} = \int_{\mathbb{M}} \omega$$

for each $\omega \in L^1(\bigwedge^n \mathbb{M})$; see e.g. [17]. Thus, by Lemma 3.2, it suffices to show that for each $s > 1$, there exists $p = p(n, K, s) > 1$ so that $f^* \omega \in L^s(\bigwedge^n \mathbb{M})$ if $\omega \in L^p(\bigwedge^n \mathbb{M})$; this is a direct consequence of the higher integrability of derivatives of quasiregular mappings.

Since f is K -quasiregular, there exists $r = r(n, K)$ so that $f \in W^{1,r}(\mathbb{M}, \mathbb{M})$; see e.g. [16, 19, 10]. In particular, $|Df| \in L^r(\mathbb{M})$. Since $J_f \leq |Df|^n$ almost everywhere, we have $J_f \in L^q(\mathbb{M})$, where $q = q(n, K) := r/n > 1$.

Thus for every $s > 1$, there exists $p = p(q(n, K), s) > 1$ so that $f^* \omega \in L^s(\bigwedge^n \mathbb{M})$ for all $\omega \in L^p(\bigwedge^n \mathbb{M})$. Indeed, fix $s > 1$, and then there exists $\beta = \beta(n, K, s) \in (0, 1)$ so that $\beta q = (s - \beta)(1 - \beta)$. Let $\omega \in L^p(\bigwedge^n \mathbb{M})$, where $p := s/\beta$. Then $\omega = u \text{vol}_{\mathbb{M}}$, where $u \in L^p(\mathbb{M})$. Since $f^* \omega = (u \circ f) J_f \text{vol}_{\mathbb{M}}$, we have

$$\begin{aligned} \int_{\mathbb{M}} (u \circ f)^s J_f^s \text{vol}_{\mathbb{M}} &= \int_{\mathbb{M}} (u \circ f)^s J_f^\beta J_f^{s-\beta} \text{vol}_{\mathbb{M}} \\ &\leq \left(\int_{\mathbb{M}} (u \circ f)^{s/\beta} J_f \text{vol}_{\mathbb{M}} \right)^\beta \left(\int_{\mathbb{M}} J_f^{(s-\beta)\frac{1-\beta}{\beta}} \text{vol}_{\mathbb{M}} \right)^{\beta/(1-\beta)} \\ &\leq \left(d \int_{\mathbb{M}} u^p \text{vol}_{\mathbb{M}} \right)^\beta \left(\int_{\mathbb{M}} J_f^q \text{vol}_{\mathbb{M}} \right)^{\beta/(1-\beta)} < \infty. \end{aligned}$$

This completes the proof. \square

4. INVARIANT CONFORMAL STRUCTURE

One of our main tools is an invariant conformal structure for the given uniformly quasiregular endomorphism. In what follows we introduce some terminology and basic observations on measurable conformal structures.

A *measurable conformal structure* G on \mathbb{M} is a bounded measurable mapping $G: \mathbb{M} \rightarrow S(\mathbb{M})$, where $S(\mathbb{M})$ is the space of positive definite symmetric bundle self-map on $T\mathbb{M}$ of determinant 1.

We say that a continuous $W_{\text{loc}}^{1,n}$ -mapping $f: \mathbb{M} \rightarrow \mathbb{M}$ is a *G -transformation* if it satisfies the *Beltrami equation*

$$(4.1) \quad D^t f(x) G(f(x)) Df(x) = J_f^{2/n}(x) G(x) \quad \text{a.e. } x \in \mathbb{M}.$$

The fundamental result of Iwaniec and Martin [11] states that every uniformly quasiregular endomorphism of \mathbb{M} is a G -transformation for some measurable conformal structure G on \mathbb{M} . More precisely, they proved the following.

Theorem 4.1 (Iwaniec-Martin [11]). *Let Γ be an abelian quasiregular semigroup of endomorphisms of \mathbb{M} . Then there exists a measurable conformal structure G_Γ on \mathbb{M} so that all mappings in Γ are G_Γ -transformations.*

While Iwaniec and Martin prove the existence of an invariant conformal structure only for abelian quasiregular semigroups of \mathbb{S}^n , their proof applies almost verbatim to all closed Riemannian manifolds \mathbb{M} . Indeed, in the proof of [11, Theorem 5.1], instead of the canonical identification $T_x \mathbb{S}^n = \mathbb{R}^n$ for $x \in \mathbb{R}^n$, where $\mathbb{R}^n \subset \hat{\mathbb{R}}^n = \mathbb{S}^n$, it suffices to use point-wise isometries $T_x \mathbb{M} \rightarrow \mathbb{R}^n$ given by some orthonormal basis of $T_x \mathbb{M}$ at $x \in \mathbb{M}$. We leave the details to the interested reader.

Let $f: \mathbb{M} \rightarrow \mathbb{M}$ be a uniformly quasiregular endomorphism. In what follows, we denote by $G = G_f$ a choice of a conformal structure $G_{\langle f \rangle}$ for f , where $\langle f \rangle$ is the abelian semigroup generated by f . Note that the conformal structure G defines a measurable Riemannian metric $\langle \cdot, \cdot \rangle_G$ on \mathbb{M} by

$$\langle v, w \rangle_G := \langle G(x)v, w \rangle$$

for almost every $x \in \mathbb{M}$ and all $v, w \in T_x \mathbb{M}$. We denote by $|v|_G$ the norm of v with respect to this inner product; we use the same notation for the induced inner products and norms for covectors. Furthermore, we denote

$$\|\xi\|_{p,G} = \left(\int_{\mathbb{M}} |\xi(x)|_G^p \text{vol}_{\mathbb{M}}(x) \right)^{1/p},$$

where ξ is a measurable form on \mathbb{M} . Note that, since $\det G \equiv 1$, the measurable volume form vol_G for this Riemannian metric coincides with $\text{vol}_{\mathbb{M}}$ almost everywhere. We note also that, for m -forms ξ on \mathbb{M} , $\|\xi\|_{n/m,G}$ is uniformly comparable to $\|\xi\|_{n/m}$, since $\text{id}: (\mathbb{M}, \langle \cdot, \cdot \rangle_G) \rightarrow (\mathbb{M}, \langle \cdot, \cdot \rangle)$ is quasiconformal.

The G -transformations f^k are conformal mappings in the metric $\langle \cdot, \cdot \rangle_G$. Indeed,

$$\begin{aligned} \langle Df(x)v, Df(x)w \rangle_G &= \langle G(f(x))Df(x)v, Df(x)w \rangle \\ &= \langle D^t f(x)G(f(x))Df(x)v, w \rangle \\ &= J_f^{2/n}(x) \langle G(x)v, w \rangle = J_f^{2/n}(x) \langle v, w \rangle_G \end{aligned}$$

for all $v, w \in T_x \mathbb{M}$ and almost every $x \in \mathbb{M}$.

The measurable Riemannian metric $\langle \cdot, \cdot \rangle_G$ defines an n -Laplace operator $\Delta_n = \Delta_{n,G}$ and, in what follows, we consider the equation $\Delta_n u = \star \mu$, where μ is a finite Borel measure on \mathbb{M} , and the pull-back of this equation under a G -transformation. The weak form of the equation $\Delta_n u = \star \mu$ reads as

$$(4.2) \quad \int_{\mathbb{M}} \langle |du|_G^{n-2} du, d\psi \rangle_G \text{vol}_{\mathbb{M}} = \int_{\mathbb{M}} \psi \, d\mu$$

for all $\psi \in C^\infty(\mathbb{M})$, or by the density of $C^\infty(\mathbb{M})$ in $W^{1,n}(\mathbb{M})$, for all $\psi \in W^{1,n}(\mathbb{M})$. We emphasize that the n -Laplace operator Δ_n has measurable coefficients and that the equation $\Delta_n u = \star \mu$ is, in fact, an \mathcal{A} -harmonic equation $d^* \mathcal{A}(du) = \star \mu$ for the bundle map $\mathcal{A}(x, \xi) = \langle G(x)\xi, \xi \rangle^{(n-2)/2} G(x)\xi$, where $\xi \in T_x^* \mathbb{M}$ and $x \in \mathbb{M}$.

A standard argument for \mathcal{A} -harmonic morphism property for quasiregular mappings shows that the equation $\Delta_n u = \star \mu$ is invariant under a G -transformation f , that is, $\Delta_n(u \circ f) = \star f^* \mu$ if u is a solution to (4.2): for, since f is a G -transformation and u is a solution to (4.2),

$$\begin{aligned} \int_{\mathbb{M}} \langle |d(u \circ f)|_G^{n-2} d(u \circ f), d\psi \rangle_G \text{vol}_{\mathbb{M}} &= \int_{\mathbb{M}} \langle |du|_G^{n-2} du, d(f_* \psi) \rangle_G \text{vol}_{\mathbb{M}} \\ &= \int_{\mathbb{M}} f_* \psi \, d\mu = \int_{\mathbb{M}} \psi \, d(f^* \mu) \end{aligned}$$

for all $u \in W^{1,n}(\mathbb{M})$ and all $\psi \in W^{1,n}(\mathbb{M})$; see [7, p. 271]. We record this G -invariance in terms of \mathcal{A} -harmonic equations.

Lemma 4.2. *Let $f: \mathbb{M} \rightarrow \mathbb{M}$ be a uniformly quasiregular endomorphism. Then there exists a bundle map $\mathcal{A}_f: T^* \mathbb{M} \rightarrow T^* \mathbb{M}$ so that for all $u \in W^{1,n}(\mathbb{M})$,*

$$f^* d^* \mathcal{A}_f(du) = d^* \mathcal{A}_f(df^* u)$$

as measures.

5. A CONSTRUCTION OF EQUILIBRIUM MEASURE

Let $f: \mathbb{M} \rightarrow \mathbb{M}$ be a uniformly K -quasiregular endomorphism of degree $d > 1$. Let G be an invariant conformal structure for f and let Δ_n be the associated measurable n -Laplace operator. Let $\mathcal{A} = \mathcal{A}_f$ be the bundle map determined by Δ_n .

Lemma 5.1. *For every $u \in W^{1,n}(\mathbb{M})$ and every $\psi \in W^{1,n}(\mathbb{M})$,*

$$(5.1) \quad \left| \int_{\mathbb{M}} \psi \star d^* \mathcal{A}(df^* u) \right| \leq \|d\psi\|_{n,G} \|du\|_{n,G}^{n-1} d^{(n-1)/n}.$$

Proof. Since f is a G -transformation,

$$\int_{\mathbb{M}} \psi \star d^* \mathcal{A}(d(f^* u)) = \int_{\mathbb{M}} \langle |d(u \circ f)|_G^{n-2} d(u \circ f), d\psi \rangle \text{vol}_{\mathbb{M}}$$

for $u \in W^{1,n}(\mathbb{M})$ and $\psi \in W^{1,n}(\mathbb{M})$. Thus,

$$\begin{aligned} \left| \int_{\mathbb{M}} \psi \star d^* \mathcal{A}(d(f^* u)) \right| &\leq \int_{\mathbb{M}} |d(u \circ f)|_G^{n-1} |d\psi|_G \\ &\leq \|d\psi\|_{n,G} \left(\int_{\mathbb{M}} |d(u \circ f)|_G^n \right)^{(n-1)/n} \\ &\leq \|d\psi\|_{n,G} \left(\int_{\mathbb{M}} (|du|_G^n \circ f) J_f \right)^{(n-1)/n} \\ &= \|d\psi\|_{n,G} \|du\|_{n,G}^{n-1} d^{(n-1)/n} \end{aligned}$$

by Hölder's inequality and the change of variables. \square

Theorem 5.2. *There exists a probability measure μ_f on \mathbb{M} such that*

$$(5.2) \quad \frac{(f^k)^* \omega}{d^k} \rightharpoonup \mu_f$$

as $k \rightarrow \infty$ whenever $\omega \in L^p(\wedge^n \mathbb{M})$ is an n -form so that $\int_{\mathbb{M}} \omega = 1$. Here $p = p(n, K) > 1$ is as in Proposition 3.1.

Proof. By Proposition 3.1, there exists a solution $u \in W^{1,n}(\mathbb{M})$ to the equation

$$(5.3) \quad d^* \mathcal{A}(du) = \star \left(\frac{f^* \omega}{d} - \omega \right).$$

Let $\psi \in C^\infty(\mathbb{M})$. By Lemmas 5.1 and 4.2,

$$\begin{aligned} \left| \int_{\mathbb{M}} \psi d \left(\frac{(f^{k+1})^* \omega}{d^{k+1}} - \frac{(f^k)^* \omega}{d^k} \right) \right| &= \frac{1}{d^k} \left| \int_{\mathbb{M}} \psi d(f^k)^* \left(\frac{f^* \omega}{d} - \omega \right) \right| \\ &= \frac{1}{d^k} \left| \int_{\mathbb{M}} \psi \star d^* \mathcal{A}(d((f^k)^* u)) \right| \\ &\leq \|d\psi\|_{n,G} \|du\|_{n,G}^{n-1} (d^k)^{(n-1)/n-1} \\ &\leq \|d\psi\|_{n,G} \|du\|_{n,G}^{n-1} d^{-k/n} \end{aligned}$$

for every $k \in \mathbb{N}$. Since $C^\infty(\mathbb{M})$ is dense in $C^0(\mathbb{M})$, the weak limit

$$\mu := \lim_{k \rightarrow \infty} \frac{(f^k)^* \omega}{d^k}$$

exists and satisfies

$$(5.4) \quad \left| \int_{\mathbb{M}} \psi d \left(\mu - \frac{(f^k)^* \omega}{d^k} \right) \right| \leq \|d\psi\|_{n,G} \|du\|_{n,G}^{n-1} \frac{d^{-k/n}}{1 - d^{-1/n}}$$

for all $\psi \in C^\infty(\mathbb{M})$ and for all $k \in \mathbb{N}$.

We show now that the measure μ does not depend on the choice of $\omega \in L^p(\wedge^n \mathbb{M})$. Let ω and ω' be n -forms in $L^p(\wedge^n \mathbb{M})$ so that

$$\int_{\mathbb{M}} \omega = \int_{\mathbb{M}} \omega' = 1.$$

Then there exists a solution $u \in W^{1,n}(\mathbb{M})$ to the equation

$$d^* \mathcal{A}(du) = \star(\omega' - \omega)$$

and, by the same computation as the above, for all $\psi \in C^\infty(\mathbb{M})$,

$$\left| \int_{\mathbb{M}} \psi d \left(\frac{(f^k)^* \omega'}{d^k} - \frac{(f^k)^* \omega}{d^k} \right) \right| \leq \|d\psi\|_{n,G} \|du\|_{n,G}^{n-1} d^{-k/n} \rightarrow 0$$

as $k \rightarrow \infty$. Since $C^\infty(\mathbb{M})$ is dense in $C^0(\mathbb{M})$, the proof is complete. \square

6. PROOF OF THEOREM 2

For the proof of Theorem 2 we need a variant of Rickman's Montel theorem ([22, Corollary IV.3.14]) in our setting; the argument is a well-known combination of a version of Rickman's Picard theorem and Zalcman's lemma; see [8].

We denote by $B^n(r)$ the open ball in \mathbb{R}^n of radius $r > 0$ and around the origin.

Proposition 6.1 (Rickman's Montel theorem). *There exists $q = q(n, K) \in \mathbb{N}$ so that for every distinct $a_1, \dots, a_q \in \mathbb{M}$, any family $\{F_j; j \in \mathbb{N}\}$ of K -quasiregular mappings $F_j: B^n \rightarrow \mathbb{M}$ omitting a_1, \dots, a_q is normal.*

Proof. By a result of Holopainen and Rickman [9], we may fix $q = q(n, K)$ so that every K -quasiregular mapping $\mathbb{R}^n \rightarrow \mathbb{M}$ omitting q points is constant.

Suppose now that $E = \{a_1, \dots, a_q\}$ is a set of q distinct points on \mathbb{M} and suppose that a family $\{F_j\}$ of K -quasiregular mappings $B^n \rightarrow \mathbb{M} \setminus E$ is not normal. Then, changing the affine coordinate if necessary, by Zalcman's lemma, there are a sequence $(F_i) \subset \{F_j\}$, a sequence $\rho_i \searrow 0$ and a sequence of points $x_i \in B^n(1/2)$ tending to the origin such that (F_i) does not converge locally uniformly but that the sequence of mappings

$$\hat{F}_i(x) := F_i(x_i + \rho_i x)$$

from $B^n((1/2)/\rho_i)$ to \mathbb{M} converges to a non-constant K -quasiregular mapping $\hat{F}: \mathbb{R}^n \rightarrow \mathbb{M}$ locally uniformly; see [8, Section 3] or [3, Section 2].

Since all \hat{F}_i omit the set E , we have a Hurwitz type result that \hat{F} also omits the set E : for, if this is not the case, then there exist $x_0 \in \hat{F}^{-1}(E)$ and a normal neighborhood Ω of x_0 . Since $\hat{F}_i|_{\Omega} \rightarrow \hat{F}|_{\Omega}$ uniformly as $i \rightarrow \infty$,

$$i(x_0, \hat{F}) = \deg(\hat{F}(x_0), \hat{F}, \Omega) = \lim_{i \rightarrow \infty} \deg(\hat{F}_i(x_0), \hat{F}_i, \Omega) = 0,$$

which is a contradiction; see [22, Section I.4] for notation and terminology.

Thus \hat{F} is constant by the choice of q . This is a contradiction. \square

Let $\mathcal{E}(f)$ the subset of all points in \mathbb{M} having a finite backward orbit under f , that is,

$$\mathcal{E}(f) = \{x \in \mathbb{M}; \# \bigcup_{k \geq 0} f^{-k}(x) < \infty\}.$$

Then any $a \in \mathcal{E}(f)$ is periodic under f and satisfies $i(a, f) = d$. By an estimate from [22, Theorem III.4.7], any $a \in \mathcal{E}(f)$ is a superattracting periodic point of f , and in particular,

$$\mathcal{E}(f) \cap \mathcal{J}(f) = \emptyset;$$

see [8, p.83]. Since the following expansion property of $\mathcal{J}(f)$ holds, by Rickman's Montel theorem, $\mathcal{E}(f)$ has the cardinality at most $q = q(n, K)$ in Proposition 6.1.

Proposition 6.2. *Let $f: \mathbb{M} \rightarrow \mathbb{M}$ be a uniformly quasiregular endomorphism. Then for every $x_0 \in \mathcal{J}(f)$ and every neighborhood Ω of x_0 (in \mathbb{M}) small enough,*

$$\bigcup_{k \in \mathbb{N}} f^k(\Omega) = \mathbb{M} \setminus \mathcal{E}(f).$$

Proof. For each $x_0 \in \mathcal{J}(f)$ and each neighborhood Ω of x_0 , the subset $\mathbb{M} \setminus \bigcup_{k \in \mathbb{N}} f^k(\Omega)$ is completely invariant under f , and is (possibly empty and) finite by Rickman's Montel theorem. Hence $\mathbb{M} \setminus \bigcup_{k \in \mathbb{N}} f^k(\Omega) \subset \mathcal{E}(f)$, that is, $\bigcup_{k \in \mathbb{N}} f^k(\Omega) \supset \mathbb{M} \setminus \mathcal{E}(f)$.

If there is $x_0 \in \mathcal{J}(f)$ such that any neighborhood Ω of x_0 satisfies $\bigcup_{k \in \mathbb{N}} f^k(\Omega) \not\subset \mathbb{M} \setminus \mathcal{E}(f)$, then from $f^{-1}(\mathcal{E}(f)) \subset \mathcal{E}(f)$, we have $x_0 \in \overline{\mathcal{E}(f)} = \mathcal{E}(f)$. This contradicts $\mathcal{E}(f) \cap \mathcal{J}(f) = \emptyset$. \square

Having Proposition 6.2 at our disposal, we close this section with a proof of Theorem 2.

Proof of Theorem 2. Let $\omega \in C^\infty(\bigwedge^n \mathbb{M})$. Let $B = B(x, 3r)$ be an open ball in $\mathbb{M} \setminus \mathcal{J}(f)$. Then $\{f^k|_B; k \in \mathbb{N}\}$ is normal, and by passing to a subsequence, we may assume that $(f^k|_B)$ converges locally uniformly to a K -quasiregular mapping $F: B \rightarrow \mathbb{M}$. Since $\int_{B(x, 2r)} F^* \omega < \infty$, by [22, Lemma VI.8.8],

$$\mu_f(B(x, r)) = \lim_{k \rightarrow \infty} \frac{1}{d^k} \int_{B(x, r)} (f^k)^* \omega \leq \lim_{k \rightarrow \infty} \frac{1}{d^k} \int_{B(x, 2r)} F^* \omega = 0.$$

Thus $\text{spt } \mu_f \subset \mathcal{J}(f)$. In particular, by $\mathcal{E}(f) \cap \mathcal{J}(f) = \emptyset$, we can choose some $a \in \text{spt } \mu_f \setminus \mathcal{E}(f)$. By $f^{-1}(\text{spt } \mu_f) \subset \text{spt } \mu_f$, Proposition 6.2 also concludes that $\mathcal{J}(f) \subset \overline{\{f^{-k}(a); k \in \mathbb{N}\}} \subset \text{spt } \mu_f$.

Suppose that μ_f has an atom $a \in \mathbb{M}$. Then $a \in \text{spt } \mu_f = \mathcal{J}(f)$. Since μ_f is invariant under f ,

$$\mu_f(\{f(a)\}) = (f_* \mu_f)(\{f(a)\}) = \mu_f(f^{-1}(f(a))) \geq \mu_f(\{a\}),$$

so $\#\{f^k(a); k \in \mathbb{N}\} < \infty$ by $\mu_f(\mathbb{M}) = 1$. Without loss of generality, we may assume that $f(a) = a$. Then since μ_f is balanced under f ,

$$\mu_f(\{a\}) = \frac{f^* \mu_f(\{a\})}{d} = \frac{i(a, f) \mu_f(\{a\})}{d},$$

which with $\mu_f(\{a\}) > 0$ implies $i(a, f) = d$. Hence $f^{-1}(a) = \{a\}$, so $a \in \mathcal{E}(f)$. This contradicts $\mathcal{E}(f) \cap \mathcal{J}(f) = \emptyset$. \square

7. APPROXIMATION OF THE EQUILIBRIUM MEASURE

Let $f: \mathbb{M} \rightarrow \mathbb{M}$ be a uniformly K -quasiregular endomorphism of degree $d > 1$.

Theorem 7.1. *The subset*

$$E(f) := \left\{ a \in \mathbb{M}; \frac{(f^k)^* \delta_a}{d^k} \not\rightarrow \mu_f \text{ as } k \rightarrow \infty \right\}$$

is of Hausdorff dimension at most $n - 1$.

Remark 7.2. We note that $\mathcal{E}(f) \subset E(f)$ since for every $a \in \mathcal{E}(f)$, any weak limit of $(f^k)^*\delta_a/d^k$ has an atom, but μ_f has no atom by Theorem 2.

Theorem 7.1 gives a characterization of μ_f as the unique balanced probability measure ν under f whose support is in $\mathbb{M} \setminus E(f)$: for, also using the Fubini theorem,

$$\begin{aligned} \int_{\mathbb{M}} \phi d\nu &= \int_{\mathbb{M}} \phi d \frac{(f^k)^*\nu}{d^k} = \int_{\mathbb{M}} \frac{(f^k)^*\phi}{d^k} d\nu = \int_{\mathbb{M}} \left(\int_{\mathbb{M}} \phi d \frac{(f^k)^*\delta_a}{d^k} \right) d\nu(a) \\ &= \int_{\mathbb{M} \setminus E(f)} \left(\int_{\mathbb{M}} \phi d \frac{(f^k)^*\delta_a}{d^k} \right) d\nu(a) \rightarrow \int_{\mathbb{M} \setminus E(f)} \left(\int_{\mathbb{M}} \phi d\mu_f \right) d\nu = \int_{\mathbb{M}} \phi d\mu_f \end{aligned}$$

as $k \rightarrow \infty$ if $\phi \in C^0(\mathbb{M})$.

For the proof of Theorem 7.1 we recall a well-known Riesz-potential estimate based on Green's functions; we refer to [2, Chapter IV §2.3] for more details.

Lemma 7.3. *For every $a \in \mathbb{M}$ and every continuous $v \in W^{1,n}(\mathbb{M})$,*

$$\left| v(a) - \int_{\mathbb{M}} v(y) \text{vol}_{\mathbb{M}} \right| \leq \int_{\mathbb{M}} \frac{|\nabla v|(y)}{|y-a|^{n-1}} \text{vol}_{\mathbb{M}}.$$

Proof. Suppose first that $v \in W^{1,n}(\mathbb{M})$ and set

$$u := v - \int_{\mathbb{M}} v(y) \text{vol}_{\mathbb{M}}.$$

For every $a \in \mathbb{M}$, by [2, Theorem 4.13] and integration by parts, we have

$$|u(a)| \leq \int_{\mathbb{M}} \langle \nabla_y G(a, y), \nabla u(y) \rangle \text{vol}_{\mathbb{M}}$$

for every $u \in C^2(\mathbb{M})$ of zero mean on \mathbb{M} . Here $G(\cdot, \cdot)$ is Green's function on \mathbb{M} . Moreover, there exists $C > 0$ so that $|\nabla_y G(y, a)| \leq C|y-a|^{1-n}$ for all distinct $a, y \in \mathbb{M}$. Thus

$$|u(a)| \leq \int_{\mathbb{M}} \frac{|\nabla u(y)|}{|y-a|^{n-1}} \text{vol}_{\mathbb{M}}$$

for every $u \in C^2(\mathbb{M})$ of zero mean on \mathbb{M} .

In the general case of $v \in W^{1,n}(\mathbb{M})$, we may approximate, by a standard convolution argument, v by C^∞ -smooth functions v_k so that $v_k \rightarrow v$ uniformly and $\|\nabla v - \nabla v_k\|_{L^1} \rightarrow 0$ as $k \rightarrow \infty$. The claim now follows. \square

A key part of the proof of Theorem 7.1 is the following capacity estimate.

Lemma 7.4. *Let $\omega = \text{vol}_{\mathbb{M}}/\text{vol}_{\mathbb{M}}(\mathbb{M})$. Then for every $\phi \in C^\infty(\mathbb{M})$, every $\varepsilon > 0$ and every $k \geq 0$, the subset*

$$K_{\varepsilon,k}(\phi) := \left\{ a \in \mathbb{M} : \left| \int_{\mathbb{M}} \phi d \frac{(f^k)^*(\delta_a - \omega)}{d^k} \right| \geq \varepsilon \right\}$$

is compact and

$$C_1(K_{\varepsilon,k}(\phi)) \leq \frac{K^{1/n} \|\nabla \phi\|_n}{\varepsilon d^{k/n}}.$$

Proof. For every $a \in \mathbb{M}$, every $\varepsilon > 0$ and $k \geq 0$, since

$$\int_{\mathbb{M}} \phi d \frac{(f^k)^*(\delta_a - \omega)}{d^k} = \frac{1}{d^k} ((f^k)_* \phi)(a) - \int_{\mathbb{M}} (f^k)_* \phi(y) d\omega,$$

the set $K_{\varepsilon,k}(\phi)$ is clearly closed, so compact.

To show the second claim, let $\varepsilon > 0$, $k \in \mathbb{N}$, and $\phi \in C^\infty(\mathbb{M})$. We may assume that $C_1(K_{\varepsilon,k}(\phi)) > 0$. Let μ be an equilibrium measure of $K_{\varepsilon,k}(\phi)$ with respect to 1-Riesz kernel on \mathbb{M} . Let $\Omega = \mathbb{M} \setminus f^k(B_{f^k})$, where B_{f^k} is the *branch set of f^k* , that is, the set where f^k fails to be a local homeomorphism.

Since \mathbb{M} is closed, we may cover \mathbb{M} with finitely many 2-bilipschitz charts. Then, applying the Vitali covering theorem locally in charts, we may cover Ω , up to a ω -nullset, with a countable collection $\{B_i\}$ of domains that are 2-bilipschitz equivalent to Euclidean n -balls and are chosen so that f^k is univalent in components of $(f^k)^{-1}(B_i)$.

We denote by g_j^i the inverses $B_i \rightarrow U_j^i$, where U_j^i is a component of $(f^k)^{-1}(B_i)$ ($j \in \{1, \dots, d^k\}$). For every $j \in \{1, \dots, d^k\}$, we define the mapping

$$g_j : \bigcup_i B_i \rightarrow \mathbb{M}$$

such that $g_j|_{B_i} = g_j^i$ for each i . Put $U_j := g_j(\bigcup_i B_i)$. Then $|\mathbb{M} \setminus \bigcup_j U_j| = 0$ and $U_j \cap U_{j'} = \emptyset$ if $j \neq j'$.

Put $v = v_\phi := (f^k)_* \phi / d^k$. Then

$$\begin{aligned} \varepsilon &\leq \int_{K_{\varepsilon,k}(\phi)} \left| \int_{\mathbb{M}} \phi d \frac{(f^k)^*(\delta_a - \omega)}{d^k} \right| d\mu(a) \\ &\leq \int_{\mathbb{M}} \left| v(a) - \int_{\mathbb{M}} v(y) \text{vol}_{\mathbb{M}} \right| d\mu(a). \end{aligned}$$

Since $f^k(B_{f^k})$ has zero measure, by Lemma 7.3 and the change of variables, for every $j \in \{1, \dots, d^k\}$,

$$\begin{aligned} \varepsilon &\leq \int_{\mathbb{M}} \left(\int_{\mathbb{M}} \frac{|\nabla v|(y)}{|a - y|^{n-1}} \text{vol}_{\mathbb{M}} \right) d\mu(a) \\ &= \int_{\mathbb{M}} \left(\int_{\Omega} \frac{|\nabla v|(y)}{|a - y|^{n-1}} \text{vol}_{\mathbb{M}} \right) d\mu(a) \\ &= \int_{\mathbb{M}} \left(\int_{U_j} \frac{|\nabla v|(f^k(x))}{|a - f^k(x)|^{n-1}} J_{f^k}(x) \text{vol}_{\mathbb{M}} \right) d\mu(a) \\ &= \int_{U_j} \left(\int_{\mathbb{M}} \frac{d\mu(a)}{|a - f^k(x)|^{n-1}} \right) |\nabla v|(f^k(x)) J_{f^k}(x) \text{vol}_{\mathbb{M}} \\ &= \int_{U_j} U^\mu(f^k(x)) |\nabla v|(f^k(x)) J_{f^k}(x) \text{vol}_{\mathbb{M}} \\ &\leq C_1(K_{\varepsilon,k}(\phi))^{-1} \int_{U_j} |\nabla v(x)| (f^k(x)) J_{f^k}(x) \text{vol}_{\mathbb{M}}. \end{aligned}$$

Here the last inequality follows from (2.2). Hence

$$\begin{aligned} C_1(K_{\varepsilon,k}(\phi)) &\leq \frac{1}{\varepsilon d^k} \sum_{j=1}^{d^k} \int_{U_j} |\nabla v(x)|(f^k(x)) J_{f^k}(x) \text{vol}_{\mathbb{M}} \\ &= \frac{1}{\varepsilon d^k} \int_{\mathbb{M}} |\nabla v(x)|(f^k(x)) J_{f^k}(x) \text{vol}_{\mathbb{M}}. \end{aligned}$$

On the other hand,

$$\begin{aligned} |\nabla v|(f^k(x)) J_{f^k}(x) &\leq \frac{1}{d^k} \sum_{j=1}^{d^k} \left| \nabla(\phi \circ g_j)(f^k(x)) \right| J_{f^k}(x) \\ &\leq \frac{1}{d^k} \sum_{j=1}^{d^k} |Dg_j|(f^k(x)) |\nabla \phi(x)|(g_j(f^k(x))) J_{f^k}(x) \\ &\leq K^{1/n} \frac{1}{d^k} \sum_{j=1}^{d^k} |\nabla \phi(x)| J_{g_j}^{1/n}(f^k(x)) J_{f^k}(x) \\ &\leq K^{1/n} \frac{1}{d^k} \sum_{j=1}^{d^k} |\nabla \phi(x)| J_{f^k}^{(n-1)/n}(x) \\ &= K^{1/n} |\nabla \phi(x)| J_{f^k}^{(n-1)/n}(x) \end{aligned}$$

for almost every $x \in \mathbb{M}$. Thus

$$\begin{aligned} \frac{1}{d^k} \int_{\mathbb{M}} |\nabla v|(f^k(x)) J_{f^k}(x) \text{vol}_{\mathbb{M}} &\leq \frac{K^{1/n}}{d^k} \int_{\mathbb{M}} |\nabla \phi(x)| J_{f^k}^{(n-1)/n}(x) \text{vol}_{\mathbb{M}} \\ &\leq \frac{K^{1/n} \|\nabla \phi\|_n}{d^k} \left(\int_{\mathbb{M}} J_{f^k}(x) \text{vol}_{\mathbb{M}} \right)^{(n-1)/n} \\ &\leq \frac{K^{1/n} \|\nabla \phi\|_n}{d^k} d^{k(n-1)/n} \\ &\leq \frac{K^{1/n} \|\nabla \phi\|_n}{d^{k/n}}, \end{aligned}$$

which completes the proof. \square

Proof of Theorem 7.1. Let $\omega = \text{vol}_{\mathbb{M}}/\text{vol}_{\mathbb{M}}(\mathbb{M})$. By Theorem 5.2, for every $a \in \mathbb{M}$, we have

$$\limsup_{k \rightarrow \infty} \left| \int_{\mathbb{M}} \phi d \left(\frac{(f^k)^* \delta_a}{d^k} - \mu_f \right) \right| = \limsup_{k \rightarrow \infty} \left| \int_{\mathbb{M}} \phi d \frac{(f^k)^* (\delta_a - \omega)}{d^k} \right|.$$

For every $\phi \in C^\infty(\mathbb{M})$, we show that for every $\epsilon > 0$, the set

$$E(\phi) := \left\{ a \in \mathbb{M} : \limsup_{k \rightarrow \infty} \left| \int_{\mathbb{M}} \phi d \frac{(f^k)^* (\delta_a - \omega)}{d^k} \right| > 0 \right\}$$

has the $(n-1+\epsilon)$ -dimensional Hausdorff measure 0. Then the proof will be complete since a countable subset of $C^\infty(\mathbb{M})$ is dense in $C(\mathbb{M})$ and (5.2) holds.

Suppose that for some $\epsilon > 0$, $\mathcal{H}^{(n-1+\epsilon)}(E(\phi)) > 0$. Then there exists a compact subset $K' \subset E(\phi)$ of positive $(n-1+\epsilon)$ -dimensional Hausdorff measure. For every $i \in \mathbb{N}$ and every $k \in \mathbb{N}$, we denote

$$K'_{1/i,k}(\phi) := \left\{ a \in K' : \left| \int_{\mathbb{M}} \phi d \frac{(f^k)^*(\delta_a - \omega)}{d^k} \right| \geq \frac{1}{i} \right\} = K' \cap K_{1/i,k}(\phi).$$

Then $K'_{1/i,k}(\phi)$ is compact and, by the monotonicity of capacity and Lemma 7.4,

$$C_1(K'_{1/i,k}(\phi)) \leq C_1(K_{1/i,k}(\phi)) \leq i \frac{K^{1/n} \|\nabla \phi\|_n}{d^{k/n}}.$$

Thus the Hausdorff $(n-1+\epsilon)$ -content $\lambda_{n-1+\epsilon}(K'_{1/i,k})$ of $K'_{1/i,k}$ is estimated as

$$\lambda_{n-1+\epsilon}(K'_{1/i,k}(\phi)) \leq \left(i \frac{K^{1/n} \|\nabla \phi\|_\infty}{\delta d^{k/n}} \right)^{\frac{n-1+\epsilon}{n-1}}$$

for every $i \in \mathbb{N}$; see [13, Corollary III.4.2]. Thus

$$\begin{aligned} \lambda_{n-1+\epsilon}(E(\phi)) &= \lambda_{n-1+\epsilon} \left(\bigcup_{i \in \mathbb{N}} \bigcap_{N \in \mathbb{N}} \bigcup_{k \geq N} K'_{1/i,k}(\phi) \right) \\ &\leq \sum_{i \in \mathbb{N}} \lim_{N \rightarrow \infty} \sum_{k \geq N} \lambda_{n-1+\epsilon}(K'_{1/i,k}(\phi)) = 0. \end{aligned}$$

On the other hand, the assumption $\mathcal{H}^{(n-1+\epsilon)}(E(\phi)) > 0$ implies that $\lambda_{n-1+\epsilon}(E(\phi)) > 0$. This is a contradiction.

Now the proof is complete. \square

8. PROOF OF THEOREM 3

We dedicate this section for the proof of the last result mentioned in Section 1, Theorem 3. In what follows, G is an invariant conformal structure for the uniformly quasiregular endomorphism f of \mathbb{M} . In particular, f is a G -transformation.

Suppose first that ψ and ϕ are C^∞ -functions on \mathbb{M} , and let $\omega \in C^\infty(\bigwedge^n \mathbb{M})$ be an n -form satisfying

$$\int_{\mathbb{M}} \omega = 1.$$

As in the proof of Theorem 5.2, let u be a solution to the equation

$$d^* \mathcal{A}_f(du) = \star \left(\frac{f^* \omega}{d} - \omega \right)$$

as in (5.3). The first step is to show that

$$(8.1) \quad \left| \int_{\mathbb{M}} \psi(\phi \circ f^k) d \left(\mu_f - \frac{(f^{k+m})^* \omega}{d^{k+m}} \right) \right| \leq C \|du\|_{n,G}^{n-1} d^{-m/n}$$

for all $k, m \in \mathbb{N}$, where C depends only on ψ , ϕ and n .

Since f is a G -transformation, for all $k \in \mathbb{N}$,

$$\begin{aligned} \|\mathrm{d}(\phi \circ f^k)\|_{n,G}^n &= \int_{\mathbb{M}} |\mathrm{d}(\phi \circ f^k)|_G^n \mathrm{vol}_{\mathbb{M}} = \int_{\mathbb{M}} (|\mathrm{d}\phi|_G^n \circ f^k) J_{f^k} \mathrm{vol}_{\mathbb{M}} \\ &= d^k \int_{\mathbb{M}} |\mathrm{d}\phi|_G^n \mathrm{vol}_{\mathbb{M}} = d^k \|\mathrm{d}\phi\|_{n,G}^n, \end{aligned}$$

where the third equality follows from the change of variables. Thus

$$\begin{aligned} \|\mathrm{d}(\psi(\phi \circ f^k))\|_{n,G} &\leq \|(\phi \circ f^k) \mathrm{d}\psi\|_{n,G} + \|\psi \mathrm{d}(\phi \circ f^k)\|_{n,G} \\ &\leq \|\phi\|_{\infty} \|\mathrm{d}\psi\|_{n,G} + \|\psi\|_{\infty} \|\mathrm{d}\phi\|_{n,G} d^{k/n} \\ &\leq C' d^{k/n} \end{aligned}$$

for all $k \in \mathbb{N}$, where $C' > 0$ depends only on ψ and ϕ . Thus, by (5.4), we have

$$\begin{aligned} \left| \int_{\mathbb{M}} \psi(\phi \circ f^k) \mathrm{d} \left(\mu_f - \frac{(f^{k+m})^* \omega}{d^{k+m}} \right) \right| &\leq \|\mathrm{d}(\psi(\phi \circ f^k))\|_{n,G} \|\mathrm{d}u\|_{n,G}^{n-1} \frac{d^{-(k+m)/n}}{1 - d^{-(k+m)/n}} \\ &\leq C' \|\mathrm{d}u\|_{n,G}^{n-1} \frac{d^{-m/n}}{1 - 2^{-2/n}}, \end{aligned}$$

which proves (8.1) by choosing $C = C'/(1 - 2^{-2/n})$.

To show (1.2), let $\varepsilon > 0$. Having (8.1) at our disposal, we may fix $m_0 > 0$ so that

$$\left| \int_{\mathbb{M}} \psi(\phi \circ f^k) \mathrm{d} \left(\mu_f - \frac{(f^{k+m})^* \omega}{d^{k+m}} \right) \right| < \varepsilon/2$$

for all $k \in \mathbb{N}$ and all $m \geq m_0$.

For each $m \geq m_0$, let us denote

$$\tau = \tau_m = \frac{(f^m)^* \omega}{d^m}$$

for brevity. By the higher integrability of quasiregular mappings, we have, as in the proof of Proposition 3.1, that $\tau \in L^p(\wedge^n \mathbb{M})$ for some $p > 1$. Since $\phi\tau \in L^p(\wedge^n \mathbb{M})$, we conclude, by Theorem 5.2, that

$$\frac{(f^k)^*(\phi\tau)}{d^k} \rightharpoonup \left(\int_{\mathbb{M}} \phi \mathrm{d}\tau \right) \mu_f$$

as $k \rightarrow \infty$. Thus

$$\begin{aligned} \int_{\mathbb{M}} \psi(\phi \circ f^k) \mathrm{d} \frac{(f^{k+m})^* \omega}{d^{k+m}} &= \int_{\mathbb{M}} \psi(\phi \circ f^k) \mathrm{d} \frac{(f^k)^* \tau}{d^k} \\ &= \int_{\mathbb{M}} \psi \mathrm{d} \frac{(f^k)^*(\phi\tau)}{d^k} \rightarrow \left(\int_{\mathbb{M}} \psi \mathrm{d}\mu_f \right) \left(\int_{\mathbb{M}} \phi \mathrm{d}\tau \right) \end{aligned}$$

as $k \rightarrow \infty$.

In particular, there exists $k_0 = k_0(m) \geq 0$ so that

$$\left| \int_{\mathbb{M}} \psi(\phi \circ f^k) \mathrm{d}\mu_f - \left(\int_{\mathbb{M}} \psi \mathrm{d}\mu_f \right) \left(\int_{\mathbb{M}} \phi \mathrm{d} \frac{(f^m)^* \omega}{d^m} \right) \right| < \varepsilon$$

if $k \geq k_0$ and $m \geq m_0$. Since

$$\frac{(f^m)^* \omega}{d^m} \rightharpoonup \mu_f$$

as $m \rightarrow \infty$, we obtain (1.2) for smooth functions ψ and ϕ on \mathbb{M} .

The equality (1.2) for $\psi, \phi \in C^\infty(\mathbb{M})$ together with standard approximation arguments yield

$$\mu_f(f^{-k}(A) \cap B) \rightarrow \mu_f(A)\mu_f(B)$$

for all Borel sets A and B in \mathbb{M} and *a fortiori* (1.2) for all $\psi, \phi \in L^2(\mu_f)$. In particular, μ_f is ergodic; see e.g. [21, Section 2.5]. The proof of Theorem 3 is complete. \square

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